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Metrics Are Clifford Algebra Involutions

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Four-dimensional space-time, all relevant inner products, and some of the groups leaving these inner products invariant are manufactured from more basic algebraic ingredients, all inside the 8-dimensional Pauli algebra $\mathscr{P}$: (1) Euclidean 3-space E^3, (2) Minkowski 4-space M^4, (3) complex 4-space $\mathbb{C}^4$, and all three metrics and all three inner products. The groups $SO(3;\mathbb{R}) \subset SO(3,1;\mathbb{R}) \subset SO(4;\mathbb{C})$ are obtained as images of twofold covering maps of subgroups of $\mathscr{P}$ or their direct product. A method of embedding $\mathscr{P}$ in the Clifford algebra $\mathscr{C}(1; n-1)$ of n-dimensional Minkowski space is given for any $n \geq 4$. Furthermore, all three groups act not only on the relevant vector spaces, but on all of $\mathscr{C}(1; n-1)$, leaving $\mathscr{P}$ setwise invariant.

1. INTRODUCTION

Algebraically, the Pauli and Dirac algebras $\mathscr{P} \subset \mathscr{D}$ are simple because they are isomorphic to 2×2 matrix rings $\mathscr{P} \cong M_2(\mathbb{C})$ and $\mathscr{D} \cong M_2(\mathbb{H}_\mathbb{R})$ over the complexes $\mathbb{C}$ and the real quaternion division ring $\mathbb{H}_\mathbb{R}$. It is well known that $M_2(\mathbb{C})$ contains $\mathbb{H}_\mathbb{R}$ and a twofold cover of $\mathrm{SO}(3, 1; \mathbb{R})$. What is important is not that each of the inner products and groups can somehow be found inside $M_2(\mathbb{C})$ separately, but that (1) they can all be found there at once, by (2) choosing an unusual embedding of Minkowski space M^4 in $\mathscr{P}$; and (3) the computationally simple and direct formulas for the group actions $\mathrm{SO}(3;\mathbb{R}) \subset \mathrm{SO}(3,1;\mathbb{R}) \subset \mathrm{SO}(4;\mathbb{C})$ in the smaller Pauli algebra $\mathscr{P}$, as opposed to the larger Dirac algebra. The systematic method using involutions to construct both metrics and the groups leaving them invariant used here in a Clifford algebra context is believed to be new. Constructions similar to the one here can be carried out in principle in any Clifford algebra, as is explicitly shown in Section 5, whereas straight $M_2(\mathbb{C})$, $\mathbb{C}^2$-spinor, or quaternionic constructions stop at dimension 8. See Dirac (1945), Edmonds (1974), Gamba (1967), and Rastall (1964).

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The canonical embedding of Euclidean space $(\mathbb{R}^3, +++)$ into $\mathscr{P}$, i.e., $(x, y, z) \to x\sigma_1 + y\sigma_2 + z\sigma_3$, singles out a distinguished embedding of $\mathbb{R}^4$ into $\mathscr{P}$; namely $(t, x, y, z) \to t \cdot 1 + x\sigma_1 + y\sigma_2 + z\sigma_3$. However, this latter embedding does not work for the above program. A similar phenomenon has already been encountered in a related construction of Euclidean space angles and directions from particles with spin 1/2 (Penrose, 1971, p. 162).

Here the point of view will be taken that even a single point in space-time, and particularly the metric, is not an *a priori* given, but rather must be built out of even more basic and primitive objects. Support for this viewpoint can be found in Marlow (1984) and Penrose (1971); in both the dominating idea is that quantum theory and the construction of space-time depend on each other. Here the primitive irreducible building blocks are to be of two kinds: elements of a Clifford algebra, and the purely algebraic operations on it ($+$, $\cdot$, and the involutions $\sim$ and $*$). To repeat the obvious, for example, to write down the equation $g = -t^2 + x^2 + y^2 + z^2$ in no way constructs the Lorentz metric from more basic ingredients; the coordinate-dependent numbers t, x, y, z are not even elements of Clifford algebras. Clifford algebras have been used before for this (e.g., Hestenes, 1961, 1984; Keller, 1984, 1985). However, with the exception of Finkelstein (1982), the idea that the basic algebraic operations themselves, and in particular the involutions, are the really fundamental entities has not been recognized or exploited.

2. CLIFFORD ALGEBRA BASICS

2.1. Let V be a vector space over the reals $\mathbb{R}$ of finite dimension $\dim_{\mathbb{R}} V = n$, and $g = g(\cdot, \cdot)$: $V \times V \to \mathbb{R}$ a symmetric, bilinear, and nonsingular (Hestenes, 1984, p. 97) function, i.e., an inner product. Let $O(V, g)$ be the group of all linear isomorphisms $T: V \to V$ such that $g(Tv, Tw) = g(v, w)$ for all $v, w \in V$; and $SO(V, g) \lhd O(V, g)$ be the subgroup of index 2 of all those T with determinant $\det T = +1$.

2.2. *Definition and Theorem.* There exists a real algebra $\mathscr{C} = \mathscr{C}(V, g)$ called the *Clifford algebra* of (V, g) such that:

(0) $\mathbb{R} \equiv \mathbb{R} \cdot 1 \subset \mathscr{C}$, $1 =$ identity element of $\mathscr{C}$.

(i) $V \subset \mathscr{C}$ canonically; every element of $\mathscr{C}$ is a sum of products of vectors of V.

(ii) For any $v, w \in V$, $vw + wv = g(v, w) \cdot 1 \in \mathscr{C}$, ($vw$, wv are ring products inside $\mathscr{C}$).

(iii) Uniqueness: $\mathscr{C}(V, g)$ is uniquely determined by (0), (i) and (ii) up to an automorphism, which is the identity on V.

2.3. Intrinsic Involutions. For any algebra, such as $\mathscr{C} = \mathscr{C}(V, g)$, an $\mathbb{R}$-linear isomorphism of order two, which either preserves or reverses multiplication is called an *involution.* Multiplication by -1 on V only extends to an involution $^*: \mathscr{C} \to \mathscr{C}$ by $(v_1 v_2 \cdots v_r)^* = (-v_1)(-v_2)\cdots(-v_r)$ for $v_i \in V$; $(\alpha - \beta)^* = \alpha^* - \beta^*$, $(\alpha\beta)^* = \alpha^*\beta^*$, $\alpha^{**} = \alpha$ for $\alpha, \beta \in \mathscr{C}$.

There exists a Unique multiplication-reversing involution which is the identity on V, obtained by writing the individual vectors in reverse order $(v_1 v_2 \cdots v_r)^C = (v_r v_{r-1} \cdots v_2 v_1)$. [For uniqueness, see Lam (1973, p. 107).]

Their composite or product defines a multiplication-reversing involution $\tilde{}: \mathscr{C} \to \mathscr{C}$ by $\tilde{\alpha} = (\alpha^C)^* = (\alpha^*)^C$. Note that $(\alpha^*)\tilde{} = \tilde{\alpha}^*$, and $\alpha^C = \tilde{\alpha}^*$. Hence, any two of $*$, C, or $\tilde{}$ are only needed. The main unique multiplication-reversing involution on a Clifford algebra such as $\mathscr{P}$, $\mathscr{D}$, or $\mathscr{C}$ will be denoted by superscripts P, D, and C, respectively.

2.4. The point of this paper is that three facts make Clifford algebras useful in physics. (1) The same uniform mathematical mechanism and theory is available for any (V, g), for any $n = \dim_{\mathbb{R}} V$, and any g. (2) Every element $T \in O(V, g)$ is given by an inner automorphism by an element $u \in \mathscr{C}(V, g)$, that is, $T(v) = u^{-1}vu$, all $v \in V$. Furthermore, u is unique up to a nonzero real scalar multiple; and $T \in SO(V, g)$ if and only if u is in the even Clifford subalgebra $u \in \mathscr{C}^+(V, g) = \mathscr{C}^+$, where $\mathscr{C}^+(V, g) = \mathscr{C}^+$ is generated by $\{vw \mid v, w \in V\}$. (3) Every Clifford algebra $\mathscr{C}(V, g)$ carries the two natural involutions $*$ and $\tilde{}$. [For 2.1-2.4, see Cassels (1978, p. 19; p. 172, Theorem 2.1; p. 176, Theorem 3.1; and p. 177, Corollary 1).]

3. EMBEDDING $(\mathbb{R}^4, -+++)$ IN PAULI ALGEBRA

The Clifford algebra of negative-definite Euclidean space is $\mathscr{C}(\mathbb{R}^3, ---) \cong \mathbb{H}_{\mathbb{R}} \oplus \mathbb{H}_{\mathbb{R}}$; while that of our usual positive-definite Euclidean space $E^3 = (\mathbb{R}^3, +++)$ is the familiar and physically significant Pauli algebra, i.e., $\mathscr{C}(E^3) \cong \mathscr{P}$. In view of the latter, Minkowski space M^4 here is defined as $M^4 = (\mathbb{R}^4, -+++)$, and hence the Dirac algebra $\mathscr{D}$ as $\mathscr{D} = \mathscr{C}(M^4) = \mathscr{C}(\mathbb{R}^4, -+++)$. However, $\mathscr{C}(\mathbb{R}^4, +---) \cong \mathscr{C}(\mathbb{R}^4, -+++) \cong M_2(\mathbb{H}_{\mathbb{R}})$.

3.1. In 2.2, take $V = (\mathbb{R}^3, +++)$. Thus, $V = \mathbb{R}\sigma_1 + \mathbb{R}\sigma_2 + \mathbb{R}\sigma_3$, where σ_1, σ_2, and σ_3 are mutually orthogonal unit vectors. By 2.2(ii), $\sigma_i^2 = 1$ and $\sigma_i\sigma_j = -\sigma_j\sigma_i$, $i \neq j$. Thus, $\mathscr{C}(\mathbb{R}^3, +++) = \mathscr{P}$. Set $\sigma_0 = 1 \in \mathscr{P}$. Write $\mathbb{R} = \mathbb{R}\sigma_0 = \mathbb{R}1 \subset \mathscr{P}$. Since the square of the central element $\sigma_1\sigma_2\sigma_3$ is $(\sigma_1\sigma_2\sigma_3)^2 = -1$, define "$\sqrt{-1}$" to mean $\sqrt{-1} \equiv \sigma_1\sigma_2\sigma_3 \in \mathscr{P}$. Here "$\mathbb{C}$" denotes the particular copy of the complexes $\mathbb{C} \equiv \mathbb{R}\sigma_0 + \mathbb{R}\sqrt{-1} \subset$ center $\mathscr{P}$.

3.2. Define $I = -\sqrt{-1}\,\sigma_1$, $J = -\sqrt{-1}\,\sigma_2$, and $K = -\sqrt{-1}\,\sigma_3$. Then $I^2 = J^2 = K^3 = -1$, $K = IJ$, and $IJ = -JI$. Thus, $\mathscr{P}$ contains the division ring of real quaternions $\mathbb{H}_{\mathbb{R}} = \mathbb{R}\sigma_0 + \mathbb{R}I + \mathbb{R}J + \mathbb{R}K \subset \mathscr{P}$. Furthermore, $\mathbb{H}_{\mathbb{R}}$ is precisely the even Clifford subalgebra $\mathscr{P}^+ = \mathbb{H}_{\mathbb{R}} \subset \mathscr{P}$.

The quaternion algebra over $\mathbb{C}$ is actually all of $\mathscr{P}$, i.e., $\mathbb{H}_{\mathbb{C}} \equiv \mathbb{C}+\mathbb{C}I+\mathbb{C}J+\mathbb{C}K = \mathscr{P}$. Hence also $\mathscr{P} = \mathbb{H}_{\mathbb{R}} + \sqrt{-1}\,\mathbb{H}_{\mathbb{R}}$, where $\sqrt{-1}\,\mathbb{H}_{\mathbb{R}}$ are the elements of odd grade.

The involutions multiply the $8 = \dim_{\mathbb{R}} \mathscr{P}$ $\mathbb{R}$-vector space basis elements of $\mathscr{P}$ by ± 1 according to Table I. The involutions $*$ and $\tilde{}$ act disjointly on $\mathbb{H}_{\mathbb{C}} = \mathbb{C}+\mathbb{C}I+\mathbb{C}J+\mathbb{C}K$. Thus, $*$ is ordinary complex conjugation $\sqrt{-1}^* = -\sqrt{-1}$ on $\mathbb{C}$, while leaving fixed $I^* = I = \sigma_3\sigma_2$, $J^* = J = \sigma_1\sigma_3$, and $K^* = K = \sigma_2\sigma_1$; whereas $\tilde{}$ acts as multiplication by -1 on $\tilde{I} = -I$, $\tilde{J} = -J$, $\tilde{K} = -K$, and leaves $\mathbb{C}$ alone, $\sqrt{-1}^{\sim} = \sqrt{-1}$.

3.3. Notation. View $\mathscr{P}$ as $\mathscr{P} = \mathbb{H}_{\mathbb{C}}$; for $a, b, A_i, B_i \in \mathbb{C}$ let $\bar{a}, \bar{A}_i \in \mathbb{C}$ be their ordinary complex conjugates. Each $\alpha, \beta \in \mathscr{P}$ is uniquely

$$\alpha = a + A_1 I + A_2 J + A_3 K \equiv (a, A_1, A_2, A_3) = a + A$$

where $a = a\sigma_0 = a \cdot 1 = (a, 0, 0, 0)$, and $A = (0, A_1, A_2, A_3)$. Similarly, $\beta = b + B$. Thus, $\alpha^* = \bar{a} + A^*$, $A^* = (0, \bar{A}_1, \bar{A}_2, \bar{A}_3)$; however $\tilde{\alpha} = a - A$. Then $(\alpha\beta)^* = \alpha^*\beta^*$, but $(\alpha\beta)^{\sim} = \tilde{\beta}\tilde{\alpha}$. Hence $\mathscr{P} \cong \mathbb{C}^4$.

Define $(A, B) = A_1B_1 + A_2B_2 + A_3B_3$, and $A \times B$ to be the ordinary cross-product of complex vectors $A, B \in \{0\} \times \mathbb{C}^3 = \mathbb{C}^3$. Clearly, $A \times B = \frac{1}{2}(AB - BA)$. Thus,

$$AB = \tfrac{1}{2}(AB + BA) + \tfrac{1}{2}(AB - BA) = (A, B) + A \times B$$

by 2.2(ii). Thus $\alpha\beta = ab + aB + bA + AB$ or

$$\alpha\beta = ab - (A, B) + aB + bA + A \times B, \qquad A \times B = \tfrac{1}{2}(AB - BA)$$

Now extend $(\cdot, \cdot)$ to all of $\mathscr{P} = \mathbb{C}^4$ by defining $(\alpha, \beta) = ab + (A, B)$.

3.4. Embedding. Define Minkowski space M^4 to be $M^4 = \sqrt{-1}\,\mathbb{R} + \mathbb{R}I + \mathbb{R}J + \mathbb{R}K \subset \mathscr{P}$, where $\mathbb{R}^4 \cong M^4$ by

$$(t, x, y, z) \to \sqrt{-1}\, t + xI + yJ + zK \in M^4$$

It is the involutions that define the physically important subspaces. Thus:

$$M^4 = \{r \in \mathscr{P} \mid \tilde{r}^* = r^{*\sim} = -r\} \tag{1}$$

Table I

Grade	$\mathscr{P}$	$*$	P	$\sim$
0	$1 = \sigma_0$	+	+	+
1	$\sigma_1 \quad \sigma_2 \quad \sigma_3$	−	+	−
2	$\sigma_3\sigma_2 \quad \sigma_1\sigma_3 \quad \sigma_2\sigma_1$	+	−	−
3	$\sigma_1\sigma_2\sigma_3 = \sqrt{-1}$	−	−	+

consists of all elements of $\mathscr{P}$ that are skew symmetric with respect to the involution $\tilde{}^*$. Euclidean space E^3 is the subspace of M^4 that is left pointwise fixed by $*$,

$$E^3 = \mathbb{R}I + \mathbb{R}J + \mathbb{R}K = \{r \in \mathscr{P} \mid r^* = r, \tilde{r}^* = \tilde{r} = -r\} \tag{2}$$

The right hand sides of (1), (2), and (3) below are well defined in any Clifford algebra whatever.

3.5. Scalar Products. For $\alpha, \beta \in \mathscr{P} = \mathbb{C}^4$ as in 3.3 the symmetric inner product $(\cdot, \cdot)$ on $\mathbb{C}^4$ reduces to the Lorentz inner product on M^4:

1. Inner or dot:

$$(\alpha, \beta) \equiv ab + (A, B) = \tfrac{1}{2}(\alpha\tilde{\beta} + \beta\tilde{\alpha}) \in \mathbb{C}$$

2. Lorentz:

$$\gamma = \sqrt{-1}\, a + A, \qquad \delta = \sqrt{-1}\, b + B \in M^4$$

$$(\gamma, \delta) \equiv (\sqrt{-1}\, a + A, \sqrt{-1}\, b + B) \equiv -ab + (A, B) = \tfrac{1}{2}(\gamma\tilde{\delta} + \delta\tilde{\gamma})$$

3. Hilbert space $\mathbb{C}^4$:

$$\langle \alpha, \beta \rangle \equiv \bar{a}b + (A^*, B) = \bar{a}b + \bar{A}_1 B_1 + \bar{A}_2 B_2 + \bar{A}_3 B_3 = \tfrac{1}{2}(\tilde{\alpha}^*\beta + \tilde{\beta}\alpha^*)$$

The associated metrics are:

(1) $(\alpha, \alpha) = a^2 + (A, A) = \alpha\tilde{\alpha} = \tilde{\alpha}\alpha$

which is positive definite only on $\mathbb{R} \cdot 1 + \mathbb{R}I + \mathbb{R}J + \mathbb{R}K$.

(2) $(\gamma, \gamma) = -a^2 + (A, A) = \tilde{\gamma}\gamma = \gamma\tilde{\gamma}$ on M^4

(3) $\langle \alpha, \alpha \rangle = |a|^2 + |A_1|^2 + |A_2|^2 + |A_3|^2 = \tfrac{1}{2}(\tilde{\alpha}^*\alpha + \tilde{\alpha}\alpha^*)$

3.6. Minkowski space $M^4 \subset \mathscr{P}$—via

$$(t, x, y, z) \to t\sigma_1\sigma_2\sigma_3 - x\sigma_2\sigma_3 + y\sigma_1\sigma_3 - z\sigma_1\sigma_2$$

—inside $\mathscr{P}$ is not merely a convenient *ad hoc* empirical formula, but has an algebraic (≡intrinsic and natural) justification:

(a) Since $IJ = K$, $JK = I$, and $KI = J$ multiply just like the usual unit vectors in $\mathbb{R}^3$, Euclidean space "E^3" clearly should be $E^3 = \mathbb{R}I + \mathbb{R}J + \mathbb{R}K \subset \mathscr{P}$.

(b) Why should time run along $\sqrt{-1}\,\mathbb{R}$ rather than $\mathbb{R} \cdot 1$? Because, for $\alpha = a + A$, $\beta = b + B \in \mathbb{R} \cdot 1 + \mathbb{R}I + \mathbb{R}J + \mathbb{R}K$, the expression $-ab + (A, B)$ *cannot* be written as sums or differences of products of $\alpha, \tilde{\alpha}, \alpha^*, \tilde{\alpha}^*, \alpha^P, \beta, \beta^P, \cdots$, etc.

4. CONSTRUCTION OF $SO(3, 1{:}\ \mathbb{R}) \subset SO(4; \mathbb{C})$ FROM $\mathscr{P}$

4.1. Any $\alpha \in \mathbb{H}_{\mathbb{C}} = \mathscr{P}$ has an inverse $\alpha^{-1} \in \mathscr{P}$ iff $\tilde{\alpha}\alpha = \alpha\tilde{\alpha} \neq 0$, in which case $\alpha^{-1} = \tilde{\alpha}/(\tilde{\alpha}\alpha)$. Thus,

$$S^3 = \{p \in \mathbb{H}_{\mathbb{R}} \mid \tilde{p}p = 1\} \subset G = \{q \in \mathbb{H}_{\mathbb{C}} \mid \tilde{q}q = 1\}$$

are multiplicative subgroups inside $\mathscr{P}$. Left and right multiplications by any $p, q \in G$ define invertible $\mathbb{C}$-linear transformations L_p, R_q, $L_pR_q : \mathbb{C}^4 \to \mathbb{C}^4$ by $L_pr = pr$, $R_qr = rq$, and $L_pR_qr = pqr$ for $r \in \mathbb{C}^4 = \mathscr{P}$. Their determinants are $\det L_p = (\tilde{p}p)^2 = +1$ and $\det R_q = (\tilde{q}q)^2 = +1$. This and other facts can be verified by taking a 4×4 complex matrix representation of the L_p, R_q, and L_pR_q (Dauns, 1982).

The group of all $\mathbb{C}$-linear transformations of $\mathbb{C}^4 \to \mathbb{C}^4$ that preserves the inner product 3.5(1) and have determinant +1 is $SO(4; \mathbb{C})$. Since $\tilde{q}q = 1$, $p\tilde{p} = 1$, and since $r\tilde{r} \in \mathbb{C}$ is a scalar, $L_pR_q \in SO(4; \mathbb{C})$, because

$$(L_pR_qr, L_pR_qr) = (prq, prq) = (prq)(prq)^{\tilde{}} = prq\tilde{q}\tilde{r}\tilde{p} = (r, r)$$

Note that $\det(L_pR_q) = (\tilde{p}p)^2(\tilde{q}q)^2 = 1$. Use of the previously mentioned 4×4 complex matrix representation shows that the exponentials of the usual 12 real Lie algebra basis elements of $SO(4; \mathbb{C})$ are all of the above required form. Thus

$$SO(4; \mathbb{C}) = \{L_pR_q \mid (p, q) \in G \times G\}$$

4.2. Let $r = \sqrt{-1}\, t + R \in M^4$, $R = xI + yJ + zK \in E^3$ with $t, x, y, z \in \mathbb{R}$. Then

$$(r, r) = r\tilde{r} = \tilde{r}r = -t^2 + x^2 + y^2 + z^2$$

The proper Lorentz group $SO(3, 1; \mathbb{R})$ is defined as all $\mathbb{R}$-linear transformations $T: M^4 \to M^4$ of $\det T = +1$ such that (i) $(Tr, Tr) = (r, r)$ for all $r \in M^4$. In our representation all group elements will act on all of $\mathscr{P}$ as $\mathbb{C}$-linear transformations, and hence (ii) $TM^4 = M^4$, which by 3.4(1) is equivalent to $(Tr)^{\tilde{}*} = -Tr$ for all $r \in M^4$.

For $q \in G$, (i) $L_qR_{\tilde{q}^*}$ automatically preserves the Lorentz metric because $L_qR_{\tilde{q}^*} \in SO(4; \mathbb{C})$. For $r \in M^4$, since $\tilde{r}^* = -r$ and $\tilde{q}^{*\tilde{}} = q^*$, we also have that (ii) $L_qR_{\tilde{q}^*}M^4 = M^4$, because $(qr\tilde{q}^*)^{\tilde{}*} = (q^*\tilde{r}\tilde{q})^* = -qr\tilde{q}^*$. Thus, $L_qR_{\tilde{q}} \in SO(3, 1; \mathbb{R}) \subset SO(4; \mathbb{C})$, and shortly we will see explicitly that every proper Lorentz transformation is of this form.

4.3. In this representation $SO(3; \mathbb{R}) \subset SO(3, 1; \mathbb{R})$ is to consist of all $\mathbb{C}$-linear maps $T: \mathbb{C}^4 \to \mathbb{C}^4$ of determinant +1 such that first (i) T acts as the identity in the first component, i.e., for $a + A \in \mathbb{C} + \mathbb{C}^3$, $T(a + A) = a + TA$; (ii) second, $T(E^3) = E^3$; and (iii) finally for any $R \in E^3 \subset M^4$, $(TR, TR) = (R, R)$.

For $p \in S^3 \subset H_{\mathbb{R}}$, $\tilde{p}^* = \tilde{p}$, and hence $L_pR_{\tilde{p}} \in SO(3, 1; \mathbb{R})$. Let $w \in \mathbb{C}$ and $R \in E^3 \subset M^4$. Then $p(w + R)\tilde{p} = w + pR\tilde{p}$, and by 3.4(2), $pR\tilde{p} \in \{0\} \times \mathbb{R}^3$. So far $L_pR_{\tilde{p}}E^3 = E^3$; however, since $L_pR_{\tilde{p}}$ is $\mathbb{C}$-linear, also (i) $L_pR_{\tilde{p}}\lfloor\{0\} \times \mathbb{C}^3\rfloor = \{0\} \times \mathbb{C}^3$ holds. Trivially,

$$(pR\tilde{p}, pR\tilde{p}) = (pR\tilde{p})(pR\tilde{p})^{\tilde{}} = R\tilde{R} = (R, R)$$

Thus, $L_pR_{\tilde{p}} \in SO(3, R)$ and $SO(3; \mathbb{R}) = \{L_pR_{\tilde{p}} \mid p \in S^3\}$; cf. 2.4(2).

4.4. For a unit vector $N \in \mathbb{R}I+\mathbb{R}J+\mathbb{R}K \subset M^4$, set $p=\cos\frac{1}{2}\theta+\sin\frac{1}{2}\theta N \in \mathcal{P}$. Then $L_pR_{\tilde{p}} \in SO(3;\mathbb{R})$ is the θ-degree counterclockwise rotation about the axis N.

The pure velocity v boost in the (positive) N direction is $L_qR_{\tilde{q}^*} \in SO(3,1;\mathbb{R})$, where

$$q=\cosh\tfrac{1}{2}\theta+\sqrt{-1}\sinh\tfrac{1}{2}\theta N \in \mathcal{P}$$

with

$$\cosh\theta = 1/[1-(v/c)^2]^{1/2}, \qquad \sinh\theta=(v/c)/[1-(v/c)^2]^{1/2}$$

Here either $c=1$ or $r=(\sqrt{-1}\,ct, x, y, z)$, where $L_qR_{\tilde{q}^*}r=qr\tilde{q}^*$.

4.5. The surjective group homomorphism $G \to SO(3,1;\mathbb{R})$, $q \to L_qR_{\tilde{q}^*}$ has kernel $\{1,-1\}$, and $G/\{1,-1\} \cong SO(3,1;\mathbb{R})$.

For any $\alpha, \beta \in \mathcal{P}$, we have $L_\alpha L_\beta = L_{\alpha\beta}$, but $R_\alpha R_\beta = R_{\beta\alpha}$. Let $G^{op} = (G^{op}, *)$ be the same group G but with opposite multiplication $\alpha * \beta \equiv \beta\alpha$. The map $G \times G^{op} \to SO(4;\mathbb{C})$, $(q,h) \to L_qR_h$ is a surjective group homomorphism with kernel $\{(1,1),(-1,-1)\}$. Thus, $(G \times G^{op})/\{(1,1),(-1,-1)\} \cong SO(4;\mathbb{C})$.

4.6. Identify the special linear group $SL(4;\mathbb{C})$ with all $\mathbb{C}$-linear transformations of $\mathcal{P} \to \mathcal{P}$ of determinant one. It contains the multiplicative subgroup $\mathrm{Aut}_{\mathbb{C}}\,\mathcal{P}$ of algebra isomorphisms of $\mathcal{P}$ leaving its center $\mathbb{C}$ elementwise fixed; $\mathrm{Aut}_{\mathbb{C}}\,\mathcal{P}$ consists precisely of the inner automorphisms. For $p \in S^3$, $p^{-1}=\tilde{p}$ and $L_pR_{\tilde{p}}$ is an inner automorphism of $\mathcal{P}$. Figure 1 is a commutative diagram of group homomorphisms.

4.7. The unitary group $U(4;\mathbb{C})$ consists of all invertible $\mathbb{C}$-linear transformations of $\mathcal{P} \to \mathcal{P}$ that leave the Hilbert space inner product 3.5(3) invariant. Define $U=\{p \in H_{\mathbb{C}} \mid \tilde{p}^*p=1\}$. For any p, $q \in U$, the identities $q^{-1}=q^P$, $\tilde{q}q=1$, and $\sqrt{-1}\,q \in U$ hold. Also, the determinant of L_pR_q is

$$\det L_p \det R_q = (\tilde{p}p)^2(\tilde{q}q)^2 \in \mathbb{C}$$

where, by 3.5(3), the absolute values of $\tilde{q}q$ and $\tilde{p}p$ are $|\tilde{p}p|^2=\langle\tilde{p}p, \tilde{p}p\rangle=1$. The $\mathbb{C}$-linear transformation L_pR_q leaves the quadratic form $\frac{1}{2}(\tilde{\alpha}^*\alpha+\tilde{\alpha}\alpha^*)$ in 3.5(3) invariant. Thus,

$$U \times U^{op} \to U(4;\mathbb{C}), \qquad (p,q) \to L_pR_q$$

is a group homomorphism with kernel $\{(1,1),(-1,-1)\}$. We have $U \cap G = S^3$, $S^3 \subset U$ is a proper containment, and $SO(3;\mathbb{R}) \subset U(4;\mathbb{C})$.

$$\begin{tikzcd}
\{1,-1\} \arrow[r, equal] \arrow[d, tail] & \{1,-1\} \arrow[d, tail] & \{(1,1),(-1,-1)\} \arrow[d, tail] \\
S^3 \arrow[r, phantom, "\subset"] \arrow[d, "2:1", "p \,\mapsto\, L_pR_{\tilde{p}}"'] & G \arrow[r, tail, two heads, "q \,\longrightarrow\, (q,\tilde{q}^*)", "\cong"'] \arrow[d, "2:1", "q \,\mapsto\, L_qR_{\tilde{q}^*}"'] & G \times G^{op} \arrow[d, "2:1"', "(q,h) \,\mapsto\, L_qR_h"] \\
SO(3;\mathbb{R}) \arrow[r, phantom, "\subset"] \arrow[d, phantom, "\cap"] & SO(3,1;\mathbb{R}) \arrow[r, phantom, "\subset"] \arrow[d, phantom, "\cap"] & SO(4;\mathbb{C}) \arrow[d, phantom, "\cap"] \\
\mathrm{Aut}_{\mathbb{C}}\mathscr{P} \arrow[r, phantom, "\subset"] & SL(4;\mathbb{C}) \arrow[r, equal] & SL(4;\mathbb{C})
\end{tikzcd}$$

Fig. 1

4.8. Unsolved Problem. Is $U(4;\mathbb{C}) = \{L_pR_q \mid (p,q) \in U \times U^{op}\}$? The conjecture is that it is not. Now view $\mathscr{P} \subset \mathscr{D}$. Is there a multiplicative subgroup $\bar{U} \subset \mathscr{D}$ with $U \subset \bar{U}$ such that $U \times U^{op}/\{(1,1),(-1,-1)\} \cong U(4;\mathbb{C})$?

5. APPLICATIONS

Recently, Clifford algebras of dimension 32 or bigger have become useful in physics (Basri and Barut, 1983; Casalbuoni and Gato, 1980; Hestenes, 1984). For this reason a family of Clifford algebras $\mathscr{C}(1; n-1) \equiv \mathscr{C}(\mathbb{R}^n; -++\cdots+)$ for $n = 4, 5, 6, \ldots$ of generalized Minkowski spaces is constructed which includes the Dirac algebra $\mathscr{D} = \mathscr{C}(1; 3)$ as the special case $n = 4$. It is then shown that the simple computational scheme of Figure 1 is available in each $\mathscr{C}(1; n-1)$. For $n = 5$, the algebra $\mathscr{C}(1, 4)$ could be potentially useful in Kaluza-Klein and other five-component theories.

5.1. For $n \geq 4$ consider on $\mathbb{R}^n$ the diagonal Minkowski metric η: $-++\cdots+$ (with one "-1" and $n-1$ "$+1$"'s). Form the real Clifford algebra $\mathscr{C}(1; n-1) \equiv \mathscr{C}(V, \eta)$, with $V = \mathbb{R}\gamma_0 + \mathbb{R}\gamma_1 + \cdots + \mathbb{R}\gamma_{n-1}$, where $\gamma_0^2 = -1$, $\gamma_1^2 = \gamma_2^2 = \cdots = \gamma_{n-1}^2 = 1$, and $\gamma_\alpha\gamma_\beta = -\gamma_\beta\gamma_\alpha$, $\alpha \neq \beta$, for $0 \leq \alpha, \beta \leq n-1$. Set $\delta = \gamma_0\gamma_1\gamma_2\cdots\gamma_{n-1}$. Then $\delta^2 = -(-1)^{n(n-1)/2}$. Only in case $\frac{1}{2}n(n-1)$ is even, define $\sqrt{-1}=\delta\in\mathscr{C}(1;n-1)$ and define "$\mathbb{C}$" to be the copy $\mathbb{C}=\mathbb{R}+\mathbb{R}\sqrt{-1}\subset\mathscr{C}(1;n-1)$. Then the following hold:

n even	center $\mathscr{C}(1;n-1)=\mathbb{R}$
n odd; $\frac{1}{2}n(n-1)$ even	center $\mathscr{C}(1;n-1)=\mathbb{R}+\mathbb{R}\sqrt{-1}=\mathbb{C}$
n odd; $\frac{1}{2}n(n-1)$ odd	center $\mathscr{C}(1;n-1)\cong\mathbb{R}\oplus\mathbb{R}$

5.2. The following elements of $\mathscr{C}(1;n-1)$ have the same multiplication table as $\mathscr{P}$: $\sigma_1=\gamma_0\gamma_1$, $\sigma_2=\gamma_0\gamma_2$, $\sigma_3=\gamma_0\gamma_3$, and $\sqrt{-1}=\sigma_1\sigma_2\sigma_3=\gamma_0\gamma_1\gamma_2\gamma_3$. Set $\mathbb{C}=\mathbb{R}+\mathbb{R}\sqrt{-1}=\mathbb{R}+\mathbb{R}\gamma_0\gamma_1\gamma_2\gamma_3$. Then

$$\mathscr{P}=\mathbb{C}+\mathbb{C}\gamma_0\gamma_1+\mathbb{C}\gamma_0\gamma_2+\mathbb{C}\gamma_0\gamma_3\subset\mathscr{C}(1;n-1)$$

is the embedding. The main multiplication-reversing involution ^{C} on $\mathscr{C}(1;n-1)$ coincides with our previous $\tilde{}$ on $\mathscr{P}$; e.g., $\sigma_i^C=(\gamma_0\gamma_i)^C=\gamma_i\gamma_0=-\sigma_i$, and $\sqrt{-1}^C=\gamma_3\gamma_2\gamma_1\gamma_0=\sqrt{-1}$.

The inner automorphism $\alpha\to\alpha^h=h^{-1}\alpha h$ of $\mathscr{C}(1;n-1)$ by the element $h=\gamma_1\gamma_2\gamma_3$ is time reversal, on $\mathscr{P}$ only, because $h^{-1}\gamma_0 h=-\gamma_0$, while $h^{-1}\gamma_i h=\gamma_i$ for $1\le i\le 3$. Furthermore, on $\mathscr{P}$ it coincides with the previous *; e.g., $\sigma_i^h=-\sigma_i$, and

$$\sqrt{-1}^h=(\gamma_0\gamma_1\gamma_2\gamma_3)^h=\gamma_0^h\gamma_1^h\gamma_2^h\gamma_3^h=-\sqrt{-1}$$

5.3. Although possibly $\mathbb{C}=\mathbb{R}+\mathbb{R}\sqrt{-1}\not\subset$ center $\mathscr{C}(1;n-1)$, certainly $\mathscr{C}(1;n-1)$ is both a left and a right $\mathbb{C}$-vector space. Thus, all of our previous groups $SO(3;\mathbb{R})\subset SO(3,1;\mathbb{R})\subset SO(4;\mathbb{C})$ now act on all of $\mathscr{C}(1;n-1)$ as $\mathbb{C}$-linear vector space isomorphisms, since $S^3\subset G\subset\mathscr{P}\subset\mathscr{C}(1;n-1)$. Moreover, $S^3\subset\mathscr{P}$ acts by inner automorphisms of $\mathscr{C}(1;n-1)$. Thus, $SO(3;\mathbb{R})\cong S^3/\{1,-1\}$ act as algebra isomorphisms of $\mathscr{C}(1;n-1)$, while $SO(3,1;\mathbb{R})$ and $SO(4;\mathbb{C})$ even on $\mathscr{P}$ do not preserve multiplication. Also, $U\subset\mathscr{P}\subset\mathscr{C}(1;n-1)$.

5.4. The Algebra $\mathscr{C}(1;4)$. By 5.1, center $\mathscr{C}(1;4)=\mathbb{C}$. Actually, $\mathbb{C}(1;4)\cong M_4(\mathbb{C})$. Since $\dim_{\mathbb{R}}\mathscr{C}(1;4)=2^5$, also $\dim_{\mathbb{C}}\mathscr{C}(1;4)=16=\dim_{\mathbb{R}}\mathscr{D}$.

Thus, there is possibly no significant increase in computational complexity in going from 4- to 5-dimensional Minkowski space, because both Clifford algebras are 16-dimensional over their respective centers; yet $\mathscr{C}(1;4)$ has the extra fifth dimension.

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